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LETTER TO THE EDITOR

Hierarchies of cofactor systems

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Abstract

We assign to any cofactor system a whole hierarchy of such systems. A sufficient condition for their complete integrability is given. The hierarchies admit construction of non-trivial integrable systems from trivial ones.

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In [2, 3] an interesting class of nonconservative Lagrangian systems called *cofactor systems* is considered. The cofactor systems naturally appear as a class of Lagrangian systems admitting quasi-Hamiltonian representation (see [2], also [4]). One of the main properties of these systems is that (under some additional assumptions) they admit involutive conservation laws. In this letter we give a necessary condition for the complete integrability of cofactor systems and assign to any of them a whole hierarchy of completely integrable cofactor systems. The construction of the hierarchies we propose is analogical to the construction of the geodesic hierarchies considered in [1, 8]. The only difference here is that the present case admits the existence of external forces. Appearing naturally in Riemannian geometry, the geodesic hierarchies give a useful tool for obtaining non-trivial integrable systems from trivial ones [1,8]. The existence of hierarchies was a crucial point proving the commutative integrability of pseudo-Riemannian geodesically (projectively) equivalent metrics [1]. It seems that the construction of the geodesic hierarchies is connected not only to the class of the systems we consider but to a rather larger class of dynamical systems. It appears also that the subject has a long history connected with the names of Levi-Civita, Liouville, Painlevé, Weyl and many others (see [1]).

By definition, the cofactor systems are Lagrangian systems of the form

$$\frac{\partial T}{\partial q^i}(q,\dot q) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot q^i}(q,\dot q) = \mu_i(q), \tag{1}$$

where the kinetic energy T is given by the quadratic form $T \stackrel{\text{def}}{=} \frac{1}{2} g_{ij}(q)\dot q^i \dot q^j$ ($g_{ij} = g_{ji}$) and the external forces μ depend only on the coordinates $q = (q^1, \ldots, q^n)$ of the configuration space. The metric tensor $g_{ij} = g_{ij}(q)$ is assumed to be non-degenerate only.

Definition 1. *A Lagrangian system (1) is called a cofactor system iff there exists a non-degenerate self-adjoint with respect to the metric g* $(1, 1)$*-type tensor field* $A^i_j \neq \mathrm{const}\, \delta^i_j$ *such that the metric tensor* g_{ij} *and the* 1*-form* μ_i *satisfy the following equations:*

$$2\nabla_k a_{ij} = \alpha_i g_{jk} + \alpha_j g_{ik} \tag{2}$$

$$\mu_i = \frac{1}{\det A} A^k_i \frac{\partial U}{\partial q^k} \tag{3}$$

where ∇ *denotes the Levi-Civita connection corresponding to the metric* g*,* $a_{ij} \overset{\text{def}}{=} g_{ik} A^k_j$*, and* U *is a smooth function of the coordinates* q*.*

Remark 1. Let us stress here that we do not assume that the metric tensor g_{ij} is positive definite or that the operators $A|_q : T_q M^n \to T_q M^n$ are diagonalizable.

Remark 2. It follows directly from equation (2) that the one-form α_j coincides with the differential of the function $\mathrm{tr}\, A = A^i_i$.

Further, we identify the cofactor systems with the triples (g, A, μ). All tensor fields we consider are C^∞-smooth on some smooth manifold M^n. Throughout the letter we use the Einstein summation convention.

A standard procedure assigns to a cofactor system (1) a dynamical system on the phase space TM^n (the tangent bundle of M^n). In invariant terms, equation (1) can be rewritten in Newton form as $\frac{\nabla \dot{q}}{\mathrm{d}t}(t) = -\vec{\mu}$, where ∇ denotes the Levi-Civita connection corresponding to the metric g and the vector field $\vec{\mu}$ has components μ^i. If the forces μ are *conservative*, i.e. $\mu = \mathrm{d}V$, then the cofactor systems coincide with the Euler–Lagrange equations corresponding to the Lagrangian function $L \overset{\text{def}}{=} \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V$. After applying the Legendre transformation corresponding to L, $p_i \overset{\text{def}}{=} g_{ik}(q)\dot{q}^k$, the cofactor systems take Hamiltonian form on T^*M^n with Hamiltonian $H = \frac{1}{2} g^{ij}(q) p_i p_j + V$.

Examples of cofactor systems (projective hierarchy). Let $\{(q^1, \ldots, q^n)\}$ be the coordinates in $\mathbb{R}^n$. Fixing an integer $k \in \mathbb{Z}$, consider on the half-plane $\boldsymbol{H}_+ \overset{\text{def}}{=} \{\bar{q} \in \mathbb{R}^n | q^n > 0\}$ the quadratic form

$$dg_k^2 \overset{\text{def}}{=} \langle \mathcal{E}(\mathcal{A}\mathcal{E})^k \, \mathrm{d}\bar{q}, \mathrm{d}\bar{q} \rangle, \tag{4}$$

where $\bar{q}$ denotes the row-vector $\bar{q} \overset{\text{def}}{=} (q^1, \ldots, q^n)$, the square matrices $\mathcal{A}$ and $\mathcal{E}$ are given by $\mathcal{A} \overset{\text{def}}{=} \bar{q}'\bar{q} + \mathrm{diag}(\epsilon_1 d_1, \ldots, \epsilon_{n-1} d_{n-1}, 0)$ ($\bar{q}'$ denotes the transposition of $\bar{q}$), $\mathcal{E} \overset{\text{def}}{=} \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_k = \pm 1$ and $d_i \neq d_j$ $(i \neq j)$ are fixed real constants. The brackets $\langle ., . \rangle$ denote the standard Euclidean scalar product in $\mathbb{R}^n$. Denote by g_k the symmetric bilinear form corresponding to the quadratic form dg_k^2. Consider the $(1, 1)$-type tensor field $A \overset{\text{def}}{=} \mathcal{A}\mathcal{E}$. The operator A is self-adjoint with respect to g_k and non-degenerate on $\boldsymbol{H}_+$. Taking $U \in C^\infty(\boldsymbol{H}_+)$, we derive μ_i from formula (3). The results proved in [1] section IV show that the triples (g_k, A, μ) are cofactor systems.

Remark 3. Indeed, it is proved in [1] (section IVA) that the metrics g_k and $d\bar{g}_k^2 \overset{\text{def}}{=} \frac{1}{q^{n2}} \langle \mathcal{E}(\mathcal{A}\mathcal{E})^{k-1} \, \mathrm{d}\bar{q}, \mathrm{d}\bar{q} \rangle$ are geodesically equivalent (see definition 1 in [1]). The operator A defined above coincides (up to a multiplication by a constant) with the operator $A(g_k, \bar{g}_k)$ given by formula (1) in [1]. Finally, lemma 1 in [1] (section IIA) shows that equation (2) from the definition of cofactor systems is satisfied.

Remark 4. The metric $\bar{g}_1$ is a metric of constant negative curvature. Other examples of cofactor systems can be extracted in the same way from the results proved in [7, 8]. It appears that the metric of the standard ellipsoid (the motion of a free particle restricted on the surface of the ellipsoid) and the metric of the Poisson sphere (and its analogues: see [7]) give examples of cofactor systems.

It was pointed out in [2] section IV that, provided μ is fixed, equation (3) is locally solvable with respect to U iff $\mathrm{D}_A\mu = 0$. The differential operator D_A acts on differential forms by the formula $\mathrm{D}_A\theta \stackrel{\text{def}}{=} (1/\det A)\,\mathrm{d}_A((\det A)\theta) = \mathrm{d}_A\theta + \alpha\wedge\theta$. Here $\mathrm{d}_A : \Omega^k(M^n) \to \Omega^{k+1}(M^n)$ denotes a first-order derivation acting on the exterior algebra $\Omega^*(M^n)$ of scalar-valued differential forms and commuting with the exterior differential $\mathrm{d} : \Omega^k(M^n) \to \Omega^{k+1}(M^n)$, i.e. we suppose that $\mathrm{d}_A(\omega_1\wedge\omega_2) = (\mathrm{d}_A\omega_1)\wedge\omega_2 + (-1)^{\deg\omega_1}\omega_1\wedge\mathrm{d}_A\omega_2$ and $\mathrm{d}\mathrm{d}_A + \mathrm{d}_A\mathrm{d} = 0$. On smooth functions the 'differential' d_A acts by the formula $\mathrm{d}_A f = A^*\,\mathrm{d}f$. The last three conditions define d_A uniquely. As mentioned in [2], equation (2) implies that the Nijenhuis torsion N_A vanishes. Therefore, $\mathrm{d}_A^2 = 0$ (see [5]).

Remark 5. Suppose that the $(1, 1)$-type tensor field J has vanishing Nijenhuis torsion $N_J = 0$. It was proved in [5] that, provided J is non-degenerate on M^n, the derivation d_J satisfies the Poincaré lemma. Moreover, Willmore proved that the cohomologies $H^*_J(M^n)$ and $H^*(M^n)$ of the differential complexes $(\Omega^*(M^n), \mathrm{d}_J)$ and $(\Omega^*(M^n), \mathrm{d})$ are isomorphic. The simple arguments used in [2] section IV show that D_A also satisfies the Poincaré lemma.

The aim of this letter is to prove the next theorem. Let k be an integer. Denote by $g^{(k)}$ the metric $g^{(k)}(X, Y) \stackrel{\text{def}}{=} g(A^kX, Y)$ (from now on we write simply $g^{(k)} = gA^k$).

Theorem 1. *Suppose that the triple (g, A, μ) is a cofactor system. Then for every integer k the triple $(g^{(k)}, A, \mu)$ is a cofactor system as well.*

Therefore, in this way, we obtain hierarchies of quasi-Hamiltonian systems from the given one (see [2] section III).

Proof of theorem 1. Equation (2) is the same as equation (8) in lemma 1 in [1] (we replace λ_j by $\alpha_j/2$). It simply means that the metrics g and $\bar{g} \stackrel{\text{def}}{=} (1/\det A)gA^{-1}$ are geodesically equivalent (see definition 1 in [1]). Consider the tensor field $A(g, \bar{g})$ given by formula (1) in [1]. A direct calculation shows that $A(g, \bar{g}) = A$. Consider the *geodesic hierarchy* corresponding to the pair metrics g and $\bar{g}$ (see section IIB in [1])

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
g^{(-1)} & \stackrel{g.e.}{\longleftrightarrow} & \bar{g}^{(-1)} \\
\downarrow & & \downarrow \\
g & \stackrel{g.e.}{\longleftrightarrow} & \bar{g} \\
\downarrow & & \downarrow \\
g^{(1)} & \stackrel{g.e.}{\longleftrightarrow} & \bar{g}^{(1)} \\
\downarrow & & \downarrow
\end{array}$$

For every integer k the metrics $g^{(k)}$ and $\bar{g}^{(k)}$ are geodesically equivalent. It can be easily checked that $A(g^{(k)}, \bar{g}^{(k)}) = A$. Therefore, by lemma 1 in [1], the quadratic form $a^{(k)} \stackrel{\text{def}}{=} g^{(k)}A$ satisfies

$$2\nabla^{(k)}_p a^{(k)}_{ij} = \alpha_i g^{(k)}_{jp} + \alpha_j g^{(k)}_{ip}, \tag{5}$$

where $\nabla^{(k)}$ is the Levi-Civita connection corresponding to the metric $g^{(k)}$. Finally, remark that equation (3) is independent of the choice of the metric. This completes the proof of theorem 1.

Following [1] we give the next definition.

Definition 2. *The sequence* $(g^{(k)}, A, \mu)$, $k \in \mathbb{Z}$, *is called geodesic hierarchy of cofactor systems corresponding to the cofactor system* (g, A, μ).

Combining the results of section VII in [2] and section IIIB in [1] we obtain the next corollary. Suppose that the manifold M^n is connected and let its first cohomology group vanish, i.e. $H^1(M^n) \cong 0$.

Theorem 2. *Let* (g, A, μ) *be a cofactor system on* M^n *where* $\mu = \mathrm{d}V$ *is the differential of a smooth function* $V \in C^\infty(M^n)$. *Suppose that there exists a point* $q_0 \in M^n$ *where the degree of the minimal polynomial of the operator* $A|_{q_0}$ *is* n. *Then for every integer* $k \in \mathbb{Z}$ *the system* $(g^{(k)}, A, \mu)$ *is completely integrable.*

Therefore, having a 'potential' cofactor system ($\mu = \mathrm{d}V$), we obtain a whole hierarchy of completely integrable systems.

Remark 6. The condition $H^1(M^n) \cong 0$ assumed in theorem 2 can be replaced by the condition that for every fixed real constant c the integrals $\int_{[\gamma]\in H_1(M^n)} \mu_c$ vanish. Here $\mu_c \stackrel{\text{def}}{=} \det(A + c\mathbf{1})(A + c\mathbf{1})^{-1*}\mu$ and $\mathbf{1}$ is the identity endomorphism on TM^n.

Remark 7. To prove the Liouville integrability of the systems considered in theorem 2 we do not need the strong condition that the eigenvalues of A are functionally independent. It is sufficient to assume only that there exists a point $q_0 \in M^n$ where the degree of the minimal polynomial of $A|_{q_0}$ is n. We do not assume that the operators $A|_q$ are diagonalizable. In the case of diagonalizable $A|_q$ a weaker statement, based on the Benenti theory of orthogonal separation of the variables, is formulated in [10].

Remark 8. Suppose that only the metric g is given. To apply theorem 2 we need to find a cofactor system (g, A, μ) such that $\mu = \mathrm{d}V$ and the degree of the minimal polynomial of $A|_{q_0}$ (q_0 is some fixed point on M^n) is n. It can be easily seen that the solutions a_{ij} (and A^i_j) of equation (2) form a linear space $\mathcal{G}(g)$ (called the *geodesic class* of the metric g). Lemma 1 in [1] assigns to any non-degenerate $A \in \mathcal{G}(g)$ a metric $\bar{g} \stackrel{\text{def}}{=} (1/\det A)gA^{-1}$ that has the same unparametrized geodesics as g. Hence, the condition of complete integrability proposed in theorem 2 can be interpreted as the condition of the existence of a 'maximally non-proportional' with respect to $T \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$ kinetic energy $\bar{T} \stackrel{\text{def}}{=} \frac{1}{2}\bar{g}_{ij}(q)\dot{q}^i\dot{q}^j$, that leads to the same (unparametrized) free motion on the configuration space as T.

Proof of theorem 2. Fixing an integer k, consider the cofactor system $(g^{(k)}, A, \mu)$. It follows from the definition of cofactor systems that $\mathrm{D}_A\mu = 0$. It is clear that $D_{\mathbf{1}}\mu = \mathrm{d}\mu = 0$. It can be easily seen that the condition $D_{A+c\mathbf{1}}\mu = 0$ ($D_{A+c} = \mathrm{D}_A + cd$) implies that the forms $\mu_c = \det(A + c\mathbf{1})(A + c\mathbf{1})^{-1*}\mu = \mu_{n-1}c^{n-1} + \cdots + \mu_0$ are closed for every value of the real parameter c. Indeed, let us fix a point $x \in M^n$ and take an arbitrary c ($|c| > N > 0$) such that the operator $A + c\mathbf{1}$ is non-degenerate in an open neighborhood $U(q)$ of q. The Poincaré lemma for the operator $D_{A+c\mathbf{1}}$ (see remark 5) gives that there exist an open neighbourhood $U'(q) \subset U(q)$ of the point q and a smooth function $\kappa \in C^\infty(U'(q))$ such that $D_{A+c\mathbf{1}}\kappa = \mu$. The last equation is equivalent to the equation $\mathrm{d}\kappa' = \det(A+c\mathbf{1})(A + c\mathbf{1})^{-1*}\mu$, where $\kappa' \stackrel{\text{def}}{=} \det(A+c\mathbf{1})\kappa$. Hence, if $|c| > N$ then μ_c is closed. This implies that any of the forms $\mu_{n-1} = \mu, \mu_{n-2}, \ldots, \mu_0$ is closed. The condition $H^1(M^n) \cong 0$ simply means that there exist smooth functions $V_{n-1} = V, V_{n-2}, \ldots, V_0 \in C^\infty(M^n)$ such that $\mathrm{d}V_k = \mu_k$ ($k = 0, \ldots, n-1$). Taking $V(c) \stackrel{\text{def}}{=} V_{n-1}c^{n-1} + \cdots + V_0$, we obtain that $\mathrm{d}V(c) = \mu_c$. Applying theorem 4, section VII, in [2] to the cofactor system $(g^{(k)}, A, \mu)$, we obtain that the one-parameter family of functions

$$
\begin{aligned}
H_c^{(k)}(p) &\stackrel{\text{def}}{=} \tfrac{1}{2}\langle \det(A + c\mathbf{1})(A + c\mathbf{1})^{-1}A^{-k}g^{-1}p, p\rangle + V(c) \\
&= I_{n-1}^{(k)}(p)c^{n-1} + \cdots + I_0^{(k)}(p), \qquad p \in T^*M^n
\end{aligned} \tag{6}
$$

are integrals in involution of the cofactor system $(g^{(k)}, A, \mu)$. The symbol g^{-1} denotes the inverse to the Legendre transformation corresponding to the metric g and the bracket $\langle ., . \rangle$ denotes the canonical pairing between TM^n and T^*M^n.

Remark 9. The coefficient $I_{n-1}^{(k)}(p) = \frac{1}{2}\langle A^{-k} g^{-1} p, p\rangle + V$, $p \in T^*M^n$, is a Hamiltonian of the cofactor system $(g^{(k)}, A, \mu)$.

Finally, let us prove that the integrals $I_{n-1}^{(k)}, \ldots, I_0^{(k)}$ are functionally independent. As we have seen the metrics $g^{(k)}$ and $\bar{g}^{(k)}$ are geodesically equivalent and $A(g^{(k)}, \bar{g}^{(k)}) = A$. By definition, the rank of the geodesic equivalence $r = r(g^{(k)}, \bar{g}^{(k)})$ coincides with the number $\max_{q \in M^n} r_A(q)$ where $r_A(q)$ denotes the degree of the minimal polynomial of the operator $A|_q$. Hence, $r = n$. Remark 3 in [1] shows that the set of points $q \in M^n$ where $r_A(q) = n$ is open and dense in M^n. Let us fix a point q where $r_A(q) = n$. Proposition 3 and item (2) of lemma 2 (both in [1], section IIIB) give that the restrictions $I_{n-1}^{(k)}|_{T_qM^n}, \ldots, I_0^{(k)}|_{T_qM^n}$ are functionally independent on the fibre $T_qM^n \hookrightarrow TM^n$. This completes the proof of theorem 2. □

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